# Inventory Mathematics and Management

## From Stock Records and EOQ to Risk-Aware Optimization and Industrial Digital Artifacts

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# Learning outcomes: from beginner to advanced

## Foundations

1. Represent physical and informational stock flows.
2. Interpret demand, lead time, variability, and cost.
3. Derive EOQ, safety stock, and reorder points.

## Advanced decision science

4. Compare service levels and replenishment policies.
5. Formulate constrained and stochastic models.
6. Validate a customer-facing digital artifact with real KPIs.

**What? Why? How? When? Where?** These questions organize the lecture.

# A business problem before a mathematical model

## Manager's question

For a given product at a particular warehouse: *how much should we order, when should we reorder, and what business cost or service outcome will result?*

**Competing objectives**

- Fewer orders and lower transaction cost.
- Less capital tied up in stock.
- Fewer stockouts and lost sales.
- Less expiry, damage, and obsolescence.

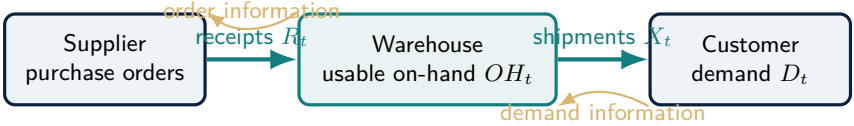
**Decision variables and information**

- Order quantity $Q$ and reorder trigger $r$.
- Demand history and forecasts.
- Supplier lead times and reliability.
- Operational constraints and costs.

## Decision insight

An inventory policy is a decision rule, not merely a numerical formula.

# Inventory as a physical and information system




## Core inventory balance

$$I_t^{\text{close}} = I_t^{\text{open}} + R_t - X_t - W_t.$$

$W_t$ represents damage, shrinkage, expiry write-offs, or other inventory adjustments.

# Inventory balance: a first worked example

## Warehouse transactions

| Transaction | Units |
|---|---|
| Opening stock | 500 |
| Receipts | $+200$ |
| Customer shipments | $-450$ |
| Damage / shrinkage | $-10$ |
| Closing stock | **240** |

## Mathematical reconciliation

$$I_t^{\text{close}} = 500 + 200 - 450 - 10 = 240.$$

A digital system should reconcile this result against physical stock counts and the transaction ledger.

## Decision insight

Inventory records must be auditable before applying a forecast or optimization algorithm.

# On-hand inventory versus inventory position

## Continuous-review policy state

$$\boxed{\text{IP} = OH + OO - B}$$

$OH$: usable on-hand units; $OO$: outstanding inbound order units; $B$: backordered demand.

## Illustration

Suppose $OH = 220$, $OO = 150$, and $B = 30$. Then

$$\text{IP} = 220 + 150 - 30 = 340 \text{ units.}$$

If the reorder point is 333 units, the *inventory position* is above the trigger: do not place another standard order yet.

## Decision insight

Ignoring outstanding orders can create duplicate orders and excessive inventory.

# Types of inventory and their mathematical role

| Type | Why it exists | Basic measure / model |
|---|---|---|
| **Cycle stock** | Ordinary demand between receipts | Mean $Q/2$ (classical EOQ) |
| **Safety stock** | Protection against uncertainty | $\mathrm{SS} = r - \mathbb{E}[X_L]$ |
| **Pipeline inventory** | Units being replenished | Mean $\bar{d}L$ (steady flow) |
| **Raw materials** | Inputs awaiting production | $I_t^{RM}$ |
| **Work-in-process** | Units undergoing transformation | $\mathbb{E}[WIP] = \text{throughput} \times \text{flow time}$ |
| **Finished goods** | Ready for customer shipment | $I_t^{FG}$ |
| **Perishable inventory** | Units with limited remaining shelf life | Batch age $a_{bt} \leq T_b$ if usable |

Work-in-process relation: Little's Law under stable-flow conditions.

# Data dictionary: start with units and definitions

| **Symbol** | Units | Meaning |
|---|---|---|
| $D$ | units/year | Expected annual demand |
| $\bar{d}$ | units/working day | Expected daily demand, $D/N$ |
| $\sigma_d$ | units/working day observation | Daily-demand standard deviation |
| $L$ | working days | Supplier replenishment lead time |
| $S$ | Rs/order | Fixed cost per order |
| $H$ | Rs/(unit · year) | Inventory holding cost |
| $c$ | Rs/unit | Unit acquisition value |
| $Q$ | units/order | Replenishment lot size |
| $\alpha$ | probability | Cycle-service target |

## Common source of errors

Do not mix calendar days and working days; do not treat a percentage carrying rate as an amount in Rs per unit-year.

# Demand: mean, variance, and coefficient of variation

## Five observed daily demands

$20,\ 25,\ 30,\ 35,\ 40$ units.

$$\bar{d} = \frac{1}{n}\sum_{t=1}^{n} d_t = 30.$$

$$s_d = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(d_t - \bar{d})^2} = \sqrt{62.5} \approx 7.91.$$

$$CV = \frac{s_d}{\bar{d}} \approx 0.264.$$

- Mean: how much demand to anticipate.
- Variation: how uncertain that demand is.
- $CV$: relative instability across SKUs.

## Decision insight

Observed sales are not necessarily true demand when stockouts prevent sales.

# Lead-time demand: a random sum

## Fixed lead time, independent identically distributed daily demand

Let $X_L = \sum_{j=1}^{L} D_j$, where $D_j$ are daily demands with mean $\bar{d}$ and variance $\sigma_d^2$.

$$\mathbb{E}[X_L] = L\bar{d}, \qquad \mathrm{Var}(X_L) = L\sigma_d^2, \qquad \sigma_L = \sigma_d\sqrt{L}.$$

## Example

At $\bar{d} = 40$ units/day, $\sigma_d = 12$ units/day, $L = 7$ working days:

$$\mathbb{E}[X_7] = 280, \qquad \sigma_7 = 12\sqrt{7} \approx 31.75.$$

# Inventory cost taxonomy

| Component | Expression | Business meaning |
|---|---|---|
| **Purchasing** | $Dc$ | Fixed acquisition price $c$ per unit |
| **Ordering** | $DS/Q$ | $D/Q$ orders at $S$ Rs each |
| **Cycle holding** | $HQ/2$ | Average cycle inventory $Q/2$ |
| **Safety holding** | $H\,\mathrm{SS}$ | Annual carrying cost of safety units |
| **Shortage penalty** | $p\,U$ | Cost of $U$ unfulfilled units, if modelled |
| **Financing** | $r_f c\bar{I}$ | Capital cost; avoid double-counting in $H$ |
| **Expiry / damage** | $\sum_b c_b W_b$ | Write-offs by batch, net of any recoveries |

Different business objectives require different terms in the objective function.

# Relevant cost is not the same as total business cost

## Classical relevant annual cost with fixed safety stock

$$\boxed{TC(Q, \mathrm{SS}) = \frac{DS}{Q} + H\left(\frac{Q}{2} + \mathrm{SS}\right).}$$

With constant unit acquisition cost, adding $Dc$ does not change the unconstrained minimizer of $Q$.

## Industrial limitation

A complete economic model may require purchase discounts, shortage penalties, spoilage, transport, borrowing, capacity, and supplier penalties. Such terms may *change the optimal order size*.

# Economic Order Quantity (EOQ): assumptions

## Decision question

What fixed lot size $Q$ balances the annual cost of ordering and holding stock?

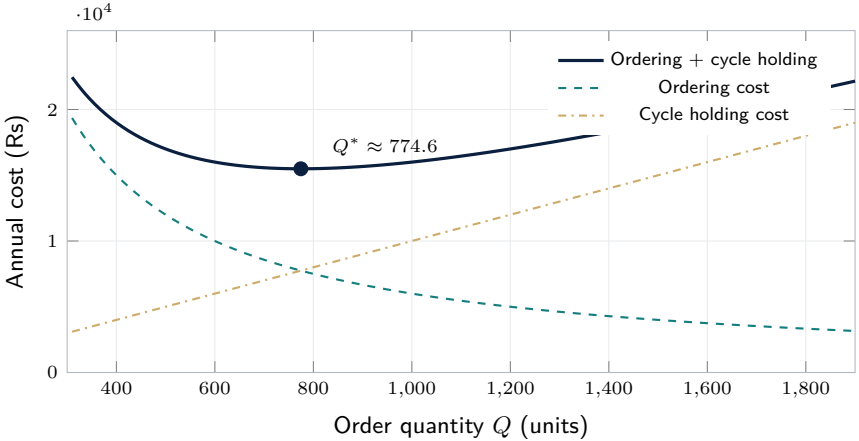
- Deterministic, constant demand rate.
- Fixed cost per replenishment order.
- Constant unit annual carrying cost.
- Instantaneous receipt of a whole lot.

## Classical objective

$$\min_{Q>0} TC(Q) = \frac{DS}{Q} + \frac{HQ}{2}.$$

- No price breaks or order-size constraints.
- No stockouts in the ideal model.
- Safety stock, if added, remains fixed as $Q$ varies.

# EOQ cost trade-off: visualize the optimum



# Deriving the EOQ formula from first principles

## Cost function

$$TC(Q) = \frac{DS}{Q} + \frac{HQ}{2}.$$

Differentiate and set the derivative equal to zero:

$$\frac{dTC}{dQ} = -\frac{DS}{Q^2} + \frac{H}{2} = 0 \quad \Longrightarrow \quad \boxed{Q^* = \sqrt{\frac{2DS}{H}}}.$$

## Why is this a minimum?

$$\frac{d^2TC}{dQ^2} = \frac{2DS}{Q^3} > 0 \qquad (D, S, Q > 0).$$

The cost function is strictly convex for positive $Q$.

# EOQ structural results and sensitivity

## At the optimum

$$\frac{DS}{Q^*} = \frac{HQ^*}{2}$$

$$TC(Q^*) = \sqrt{2DSH}.$$

## How far from optimal?

Write $Q = xQ^*$. Then

$$\frac{TC(xQ^*)}{TC(Q^*)} = \frac{x + x^{-1}}{2}.$$

- $x = 0.9$: cost increases $0.556\%$.
- $x = 1.1$: cost increases $0.455\%$.

## Decision insight

A policy can remain economically close to EOQ even when practical carton or pallet constraints change $Q$ slightly.

# From theoretical EOQ to a whole-unit purchase order

## Operationally feasible order size

For indivisible units, compare the adjacent candidates $\lfloor Q^* \rfloor$ and $\lceil Q^* \rceil$ (subject to $Q \geq 1$), then choose the less costly one.

$$Q^*_{\text{integer}} = \underset{Q \in \{\max(1, \lfloor Q^* \rfloor),\ \max(1, \lceil Q^* \rceil)\}}{\arg\min} \left\{ \frac{DS}{Q} + \frac{HQ}{2} \right\}.$$

## If suppliers require full cartons

With pack size $m$, impose $Q = km$, $k \in \mathbb{Z}_{\geq 1}$. If minimum order quantity $M$ also applies, impose $Q \geq M$.

# Deterministic reorder point: when to order?

## Fixed lead time and constant daily demand

$$\boxed{r = \bar{d}L}$$

When *inventory position* reaches $r$, order the chosen replenishment quantity.

## Numerical check

If demand is 40 units/working day and lead time is 7 working days,

$$r = 40(7) = 280 \text{ units.}$$

A reorder at 280 units exactly covers expected lead-time consumption; it provides *no uncertainty buffer.*

# Safety stock: moving from certainty to probability

## Normal approximation with independent daily demand and fixed $L$

$$\boxed{\mathrm{SS} = z_\alpha \sigma_d \sqrt{L}}, \qquad \boxed{r = \bar{d}L + \mathrm{SS}}.$$

$z_\alpha = \Phi^{-1}(\alpha)$ is the standard normal quantile; $\alpha$ is the target *cycle-service probability*.

- Mean demand determines the baseline stock.
- Standard deviation quantifies exposure to fluctuations.
- Service target determines the size of the buffer.

### Boundary of the formula

Not exact for correlated daily demand, erratic lead times, severe skewness, or strongly intermittent demand.

# Cycle service level: interpreting $\alpha$ and $z_\alpha$

| Target $\alpha$ | $z_\alpha$ |
|---|---|
| 90% | 1.282 |
| 95% | 1.645 |
| 97.5% | 1.960 |
| 99% | 2.326 |
| 99.9% | 3.090 |

**Formal definition**

$$CSL = \Pr(X_L \le r).$$

Higher CSL leads to higher safety stock when the uncertainty distribution is unchanged.

**Decision insight**

A 95% cycle service level means roughly 95% probability of no stockout within a replenishment cycle under the model—not 95% units filled.

# Cycle service level is not the unit fill rate

## Two different business measures

**Cycle Service Level (CSL):** probability of no shortage during a replenishment cycle:

$$CSL = \Pr(X_L \leq r).$$

**Unit Fill Rate ($\beta$):** fraction of demanded units supplied immediately from stock:

$$\beta = \frac{\text{units immediately fulfilled}}{\text{total demanded units}}.$$

## Standard continuous-review $(Q, r)$ model with backorders

$$\beta \approx 1 - \frac{\mathbb{E}[(X_L - r)^+]}{Q}, \qquad x^+ = \max(x, 0).$$

This approximation converts expected lead-time shortage into the fraction of units not filled directly from stock, under its usual model assumptions.

# The practical reorder decision: inventory position

## Reorder rule

$$\mathrm{IP} = OH + OO - B, \qquad \boxed{\text{Place a replenishment order if } \mathrm{IP} \le r.}$$

## State A

$OH = 220$, $OO = 150$, $B = 30$.

$$\mathrm{IP} = 340 > 333.$$

**No new standard order.**

## State B

$OH = 210$, $OO = 100$, $B = 20$.

$$\mathrm{IP} = 290 \le 333.$$

**Trigger replenishment.**

Actual production implementations must also consider already-triggered order events, approvals, and minimum order rules.

# Worked example: input data from IJ Artifact Lab

| Input | Symbol | Value |
|---|---|---|
| Annual demand | $D$ | $12{,}000$ units |
| Fixed ordering cost | $S$ | Rs 500/order |
| Unit annual holding cost | $H$ | Rs 20/unit-year |
| Demand-bearing working days | $N$ | $300$ |
| Standard deviation of daily demand | $\sigma_d$ | $12$ units/day observation |
| Fixed supplier lead time | $L$ | 7 working days |
| Target cycle service level | $\alpha$ | $95\%$ |

**Question**

What quantity should be ordered, when should it be ordered, and what annual relevant cost does that policy imply?

# Worked example: EOQ, order frequency, cycle duration

## Step 1: order quantity

$$Q^* = \sqrt{\frac{2(12000)(500)}{20}} = \sqrt{600000} \approx 774.60.$$

The cost-minimizing whole-unit quantity is $Q = 775$ units.

## Orders per year

$$n = \frac{D}{Q} = \frac{12000}{775} \approx 15.48.$$

## Expected working days between orders

$$T_d = \frac{NQ}{D} = \frac{300(775)}{12000} \approx 19.38.$$

# Worked example: safety stock and reorder point

$$\bar{d} = \frac{12000}{300} = 40,$$

$$\mathbb{E}[X_L] = 40(7) = 280,$$

$$\sigma_L = 12\sqrt{7} \approx 31.75,$$

$$\mathrm{SS} = 1.645(12\sqrt{7}) \approx 52.22 \longrightarrow 53 \text{ units},$$

$$r = 280 + 52.22 \approx 332.22 \longrightarrow 333 \text{ units}.$$

## Decision insight

For a 95% target cycle service level: order 775 units when inventory position is 333 units or lower.

# Worked example: annual cost decomposition

| Annual cost component | Rs |
|---|---|
| Ordering $\frac{12000 \cdot 500}{775}$ | 7,741.94 |
| Cycle holding $\frac{775}{2} \cdot 20$ | 7,750.00 |
| Safety holding $53 \cdot 20$ | 1,060.00 |
| **Relevant policy cost** | 16,551.94 |

## Explicit exclusions

- Purchase cost $Dc$.
- Shortage and lost sales.
- Finance not already included in $H$.
- Expiry, logistics, capacity effects.

## Interpretation

Rs 16,551.94 is an *estimated annual ordering and holding cost*, not the company's entire annual inventory bill.

# Time units and assumptions: common implementation errors

1. **Calendar versus working days:** use one convention for $\bar{d}$, $\sigma_d$, and $L$.
2. **Observed sales versus demand:** stockout-censored sales may understate true demand.
3. **On-hand versus inventory position:** outstanding replenishments matter.
4. **Cycle service versus fill rate:** different denominators, different customer commitments.
5. **Total cost label:** identify exactly which cost components are included.
6. **Numerical robustness:** reject missing, negative, infinite, and inconsistent inputs.

## Decision insight

A deployable inventory artifact must be mathematically correct *and* operationally unambiguous.

# Random lead times: a more realistic safety-stock model

## Assumptions: $L$ is random, independent of i.i.d. daily demand

Use the laws of total expectation and variance for random lead-time demand $X_L$:

$$\begin{aligned}\mathbb{E}[X_L] &= \bar{d}\,\mathbb{E}[L],\\ \operatorname{Var}(X_L) &= \mathbb{E}[\operatorname{Var}(X_L \mid L)] + \operatorname{Var}(\mathbb{E}[X_L \mid L])\\ &= \sigma_d^2 \mathbb{E}[L] + \bar{d}^{\,2} \operatorname{Var}(L).\end{aligned}$$

## Normal-approximation reorder point

$$\boxed{r \approx \bar{d}\mathbb{E}[L] + z_\alpha \sqrt{\sigma_d^2 \mathbb{E}[L] + \bar{d}^{\,2} \operatorname{Var}(L)}}.$$

# When the standard normal safety-stock formula may fail

## Demand complications

- Strong seasonality or promotional spikes.
- Intermittent / zero-inflated demand.
- Correlated daily demand.
- Censored history caused by stockouts.

## Supply complications

- Random lead time and partial receipts.
- Supplier disruptions or holidays.
- Multiple suppliers and shipment modes.
- Demand/lead-time dependence.

**Alternative:** use empirical lead-time demand distributions, simulation, or robust / stochastic optimization.

# Demand forecasting: basic models

## Moving average over the last $k$ periods

$$\widehat{D}_{t+1} = \frac{1}{k} \sum_{j=0}^{k-1} D_{t-j}.$$

## Simple exponential smoothing

$$\widehat{D}_{t+1} = \gamma D_t + (1-\gamma)\widehat{D}_t, \qquad 0 < \gamma \leq 1.$$

Recent observations receive more weight when $\gamma$ is larger.

## Decision insight

Forecasts should be trained on past data and evaluated on later, previously unseen periods.

# Forecast-error metrics: do not confuse accuracy with utility

Let $e_t = D_t - \widehat{D}_t$ and suppose there are $n$ evaluation periods.

$$ME = \frac{1}{n}\sum_{t=1}^{n} e_t \qquad \text{(signed bias)},$$

$$MAE = \frac{1}{n}\sum_{t=1}^{n} |e_t| \qquad \text{(absolute error)},$$

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n} e_t^2} \qquad \text{(large-error sensitivity)},$$

$$WAPE = 100\frac{\sum_t |e_t|}{\sum_t D_t} \qquad \text{(if total demand } > 0\text{)}.$$

## Decision insight

A lower forecast error is useful only insofar as it improves the downstream replenishment cost or service outcome.

# ABC–XYZ: prioritize products before optimizing them

## ABC: economic importance

$$V_i = D_i c_i.$$

Rank products by annual consumption value $V_i$ and examine cumulative shares.

- A: higher-value items.
- B: medium-value items.
- C: lower-value items.

## XYZ: demand variability

$$CV_i = \frac{\sigma_i}{\mu_i}.$$

- X: relatively stable demand.
- Y: intermediate variability.
- Z: irregular or volatile demand.

Thresholds are organizational conventions; intermittent demand needs additional diagnostics beyond CV alone.

# Replenishment policies: continuous versus periodic review

| Policy | Decision rule | Typical application |
|---|---|---|
| **Continuous** $(Q, r)$ | Order $Q$ when $\mathrm{IP} \leq r$ | Real-time monitored stock |
| **Continuous** $(s, S)$ | If $\mathrm{IP} \leq s$, order $S - \mathrm{IP}$ | Variable order quantities |
| **Periodic order-up-to** | At review times, order $\max(0, S - \mathrm{IP})$ | Scheduled weekly or monthly ordering |
| **Periodic** $(T, s, S)$ | Review every $T$; order toward $S$ if $\mathrm{IP} \leq s$ | Scheduled supplier interaction |

**Important**

Policies define *when* an order may be placed and *how much* to request; they do not automatically guarantee capacity or cost feasibility.

# Periodic review: the protection interval changes

## Review every $T$ days; fixed replenishment lead time $L$

An order-up-to level $S$ generally protects demand over the review-plus-lead-time interval:

$$\boxed{S \approx \bar{d}(T + L) + z_\alpha \sigma_d \sqrt{T + L}}.$$

At the review time, recommend $q = \max(0, S - \mathrm{IP})$, subsequently respecting packs, MOQs, budget, and capacity.

## Comparison with continuous review

Continuous review protects roughly $L$ days. Periodic review protects $T + L$ days because the system may need to wait for its next review.

# Economic Production Quantity (EPQ)

## Production is gradual rather than instantaneous

Let $d$ be demand rate, $P$ the production rate in the same time unit, with $P > d$.

$$\boxed{Q^*_{EPQ} = \sqrt{\frac{2DS}{H(1 - d/P)}}}.$$

- During production, inventory increases at net rate $P - d$.
- Maximum cycle stock is $Q(1 - d/P)$.
- Average cycle stock is $\frac{Q}{2}(1 - d/P)$.

## Decision insight

EOQ suits instantaneous replenishment; EPQ is a better baseline for in-house production where output accumulates over time.

# Newsvendor model: single-period uncertain demand

## When does this arise?

One opportunity to order before uncertain demand occurs: fresh food, event merchandise, limited campaigns, short-life products.

Define $C_u$ as the *underage cost* per unit and $C_o$ as the *overage cost* per unit. If demand has cumulative distribution $F$, then

$$\boxed{Q^* = F^{-1}\left(\frac{C_u}{C_u + C_o}\right).}$$

## Critical fractile

If $C_u = 60$ and $C_o = 40$ Rs/unit, the target demand quantile is $60/(60+40) = 0.60$. Choose the 60th percentile of the demand distribution.

# Multi-SKU inventory: coupling through constraints

For SKUs $i = 1, \ldots, n$, choose replenishment quantities $q_i$.

## Illustrative constrained model

$$\min_{\mathbf{q}} \quad \sum_{i=1}^{n} C_i(q_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} c_i q_i \le B \qquad \text{procurement budget,}$$

$$\sum_{i=1}^{n} v_i q_i \le V_{\max} \qquad \text{space capacity,}$$

$$q_i = m_i k_i, \quad k_i \in \mathbb{Z}_{\ge 0} \qquad \text{case-pack restriction.}$$

$v_i$ is storage volume per unit; $m_i$ is the pack size. MOQ and service constraints can be added.

# Inventory dynamics across time and warehouses

## Illustrative discrete-time state equation

For product $i$, warehouse $\ell$, time $t$:

$$I_{i\ell,t+1} = I_{i\ell,t} + R_{i\ell,t} - X_{i\ell,t} - W_{i\ell,t}.$$

Replenishment receipts may be delayed by lead time; inter-warehouse transfers add incoming and outgoing transfer terms.

## Operational feasibility

- $I_{i\ell,t} \geq 0$ for usable inventory.
- Warehouse capacities restrict stored units or cubic volume.
- Supplier/order-calendar restrictions limit feasible replenishments.
- Service targets link inventory decisions to uncertain demand.

# Perishable inventory: batches, age, and FEFO

## Batch-specific inventory

Each batch $b$ has:

- manufacture / receipt date;
- remaining quantity $I_{b,t}$;
- expiry time $T_b$;
- age $a_{b,t}$.

Only non-expired batches may be shipped:

$$x_{b,t} = 0 \quad \text{if } a_{b,t} > T_b.$$

## First Expired, First Out

FEFO prioritizes issuing batches with the earliest expiry, subject to product, traceability, and quality constraints.

## Cost consequence

Unsold, expired quantities generate write-offs; classical EOQ does not model these dynamics.

# Inventory performance: operations and finance

| Measure | Formula | Interpretation |
|---|---|---|
| **Inventory turnover** | $COGS/\overline{V}_I$ | Annual stock rotations |
| **Days Inventory Outstanding** | $365\overline{V}_I/COGS$ | Days capital stays in inventory |
| **Days of supply** | $OH/\bar{d}$ | Approximate current coverage |
| **Stockout-period rate** | #stockout periods/#observed periods | Frequency of stockouts |
| **Cash Conversion Cycle** | $CCC = DIO + DSO - DPO$ | Working-capital timing |
| **Inventory carrying value** | $\sum_i c_i I_i$ | Cost-valued inventory asset |

$\overline{V}_I$ denotes average cost-valued inventory; COGS, DSO and DPO must use consistent periods.

# Risk: expected cost versus adverse outcomes

## Stochastic inventory policy

Let $\omega$ represent uncertain demand, lead time, and other disturbances; let $\pi$ be a feasible inventory policy.

$$\min_{\pi \in \mathcal{P}} \mathbb{E}[C(\pi, \omega)]$$

minimizes *average* cost but may expose the business to severe shortages.

## Risk-aware alternative

$$\boxed{\min_{\pi \in \mathcal{P}} \left\{ \mathbb{E}[C(\pi, \omega)] + \lambda \, \mathrm{CVaR}_{\alpha}(C(\pi, \omega)) \right\}.}$$

$\lambda \geq 0$ controls emphasis on costly tail events; $\alpha$ chooses the upper cost tail.

# Value at Risk and Conditional Value at Risk

## Cost-tail definitions

For random cost $C$ with distribution function $F_C$:

$$\mathrm{VaR}_\alpha(C) = F_C^{-1}(\alpha).$$

A widely used optimization representation of upper-tail CVaR is

$$\boxed{\mathrm{CVaR}_\alpha(C) = \min_{\eta \in \mathbb{R}} \left[ \eta + \frac{1}{1-\alpha} \mathbb{E}[(C-\eta)^+] \right].}$$

## Interpretation

At $\alpha = 0.95$, CVaR summarizes the average of the worst 5% cost outcomes, with an appropriate tail convention when the distribution has point masses.

# How stochastic optimization becomes computable

## Scenario-based Sample Average Approximation (SAA)

Generate scenarios $\omega_1, \ldots, \omega_M$ representing plausible demand and lead-time paths.

$$\min_{\pi,\eta} \left\{ \frac{1}{M} \sum_{s=1}^{M} C(\pi, \omega_s) + \lambda \left[ \eta + \frac{1}{(1-\alpha)M} \sum_{s=1}^{M} (C(\pi, \omega_s) - \eta)^+ \right] \right\}.$$

- Enforce all budget, capacity and operational constraints on the feasible policy.
- Preserve the chronology of demand and replenishment in simulation.
- Evaluate selected policies on independent *out-of-sample* scenarios.

# An industrial case: FMCG warehouse replenishment

## Illustrative problem definition

A distributor manages 80 SKUs across four locations and seeks to reduce stockouts, excess holding costs, and expiry losses while protecting sales service.

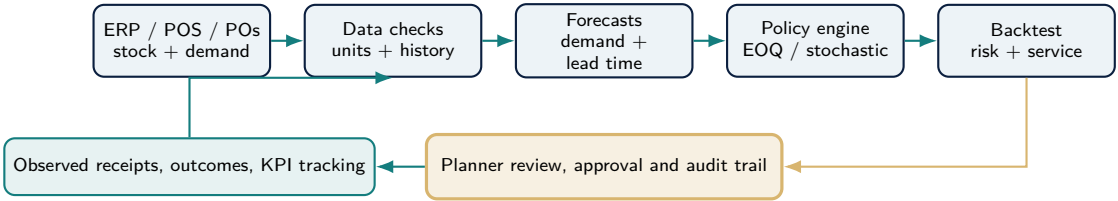
**Required data**

- SKU, prices, costs and pack sizes.
- Daily demand/sales and stockouts.
- POs, receipts, actual lead times.
- Batch age, expiry, returns.
- Supplier, budget and space constraints.

**Decisions / measurable outcomes**

- Order quantity and reorder timing.
- Supplier selection / split orders.
- Service level and realized shortages.
- Holding, lost sales and write-offs.
- Capital required and achieved savings.

# Digital artifact architecture: mathematics into operations




## Engineering principle

Expose assumptions, data provenance, recommended actions, reason codes, confidence or risk measures, and historical policy performance in one auditable workflow.

# Practical investigation: what must be verified first?

1. **Customer and decision owner:** who orders each SKU, and what can they change?
2. **Data fidelity:** can sales, physical stock, purchase orders, and receipts be reconciled?
3. **Demand:** is it seasonal, intermittent, promoted, or censored by stockouts?
4. **Supply:** fixed or random lead times, partial fills, supplier disruptions?
5. **Economics:** order/setup, carrying, finance, shortage, expiry, and logistics costs?
6. **Constraints:** MOQ, carton sizes, credit, expiry, space, vehicle capacity?
7. **Service:** CSL, unit fill rate, stockout days, or contractual service targets?
8. **Model selection:** is EOQ a sensible baseline, or is a different policy required?

# Validation: measure business impact, not only model accuracy

## Benchmark against the current policy

Compare existing ordering practice, classical EOQ/ROP, and one or more enhanced policies on the *same* historical or simulated scenarios and constraints.

**Operational KPIs**

- Realized unit fill rate and CSL.
- Stockout days / units short.
- Average inventory and turns.
- Expiry and damage write-offs.

**Financial and implementation KPIs**

- Total relevant and realized costs.
- Working capital utilization.
- Policy stability and exception count.
- Planner acceptance, overrides, time saved.

## Decision insight

Use chronological backtesting, explicit counterfactual assumptions, a pilot measurement plan, and documented limitations.

# Class discussion and exercises

1. A warehouse begins with 500 units, receives 200, ships 450, and loses 10. Find closing stock and identify the database transactions needed.
2. Derive EOQ and prove that the stationary point is a minimum.
3. For $D = 12000$, $S = 500$, $H = 20$, compare total relevant cost at $Q = 700$, $775$, and $900$.
4. For $\bar{d} = 40$, $\sigma_d = 12$, $L = 7$, calculate safety stock and reorder point at both 95% and 99% CSL.
5. Explain why $OH = 210$, $OO = 100$, $B = 20$ triggers an order at $r = 333$.
6. Suppose lead time becomes random. What new information is needed to estimate $\mathrm{Var}(X_L)$?
7. Design a two-month pilot measuring whether the proposed inventory policy delivers customer value.

# Lecture summary: from mathematics to customer value

## Core equations

$$
\begin{aligned}
I_{t+1} &= I_t + R_t - X_t - W_t, \\
Q^* &= \sqrt{2DS/H}, \\
\mathrm{SS} &= z_\alpha \sigma_d \sqrt{L}, \\
r &= \bar{d}L + \mathrm{SS}, \\
\mathrm{IP} &= OH + OO - B.
\end{aligned}
$$

## Development sequence

1. Validate inventory and demand data.
2. Select assumptions and a baseline model.
3. Optimize under real constraints.
4. Backtest against current practice.
5. Deploy with human approval.
6. Measure customer outcomes.

# Further reading and practical follow-up


- E. A. Silver, D. F. Pyke, and R. Thomas, *Inventory and Production Management in Supply Chains*.
- P. H. Zipkin, *Foundations of Inventory Management*.
- L. V. Snyder and Z.-J. M. Shen, *Fundamentals of Supply Chain Theory*.
- S. Nahmias and T. L. Olsen, *Production and Operations Analysis*.


## Recommended next practical session

Take one real SKU and reconstruct the last 90–180 days of demand, receipts, stockouts, and purchase orders. Evaluate whether classical EOQ/ROP is defensible before deploying any replenishment recommendation.